



## Robust performance hypothesis testing with the Sharpe ratio

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### ARTICLE INFO

#### Article history:

Received 7 May 2007

Received in revised form 16 March 2008

Accepted 18 March 2008

Available online 26 March 2008

#### JEL classification:

C12

C14

C22

#### Keywords:

Bootstrap

HAC inference

Sharpe ratio

### ABSTRACT

Applied researchers often test for the difference of the Sharpe ratios of two investment strategies. A very popular tool to this end is the test of Jobson and Korkie [Jobson, J.D. and Korkie, B.M. (1981). Performance hypothesis testing with the Sharpe and Treynor measures. *Journal of Finance*, 36:889–908], which has been corrected by Memmel [Mommel, C. (2003). Performance hypothesis testing with the Sharpe ratio. *Finance Letters*, 1:21–23]. Unfortunately, this test is not valid when returns have tails heavier than the normal distribution or are of time series nature. Instead, we propose the use of robust inference methods. In particular, we suggest to construct a studentized time series bootstrap confidence interval for the difference of the Sharpe ratios and to declare the two ratios different if zero is not contained in the obtained interval. This approach has the advantage that one can simply resample from the observed data as opposed to some null-restricted data. A simulation study demonstrates the improved finite sample performance compared to existing methods. In addition, two applications to real data are provided.

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### 1. Introduction

Many applications of financial performance analysis are concerned with the comparison of the Sharpe ratios of two investment strategies (such as stocks, portfolios, mutual funds, hedge funds, or technical trading rules). Since the true quantities are not observable, the Sharpe ratios have to be estimated from historical return data and the comparison has to be based on statistical inference, such as hypothesis tests or confidence intervals.

It appears that the *status quo* in the applied literature is the test of Jobson and Korkie (1981) and its corrected version by Memmel (2003); for example, see DeMiguel et al. (in press), DeMiguel and Nogales (2007), and Gasbarro et al. (2007), among others. Unfortunately, this test is not valid when returns have tails heavier than the normal distribution or are of time series nature. The former is a quite common, and by now well-known, property of financial returns. As far as the latter is concerned, serial correlation of the actual returns is, arguably, only a minor concern for stocks and mutual funds, but it is certainly relevant to hedge funds; for example, see Brooks and Kat (2002) and Malkiel and Saha (2005). However, even stocks and mutual funds often exhibit correlation of the squared returns, that is, volatility clustering; for example, see Campbell et al. (1997, Chapter 12) and Alexander (2001, Chapter 4).

In this paper, we discuss inference methods that are more generally valid. One possibility is to compute a HAC standard error<sup>3</sup> for the difference of the estimated Sharpe ratios by the methods of Andrews (1991) and Andrews and Monahan (1992), say. Such an approach works asymptotically but does not always have satisfactory properties in finite samples. As an improved alternative, we suggest a studentized time series bootstrap.

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<sup>1</sup> The views expressed herein are those of the author and do not necessarily reflect or represent those of Credit Suisse.

<sup>2</sup> The research has been supported by the University Research Priority Program "Finance and Financial Markets", University of Zurich, and by the Spanish Ministry of Science and Technology and FEDER, grant MTM2006-05650.

<sup>3</sup> In this paper, a standard error of an estimator denotes an estimate of the true standard deviation of the estimator.

It has been argued that for certain applications the Sharpe ratio is not the most appropriate performance measure; e.g., when the returns are far from normally distributed or autocorrelated (which happens for many hedge funds) or during bear markets. On the other hand, there is recent evidence that the Sharpe ratio can result in almost identical fund rankings compared to alternative performance measures; e.g., see [Eling and Schuhmacher \(2007\)](#). We do not enter this debate. Instead, we believe that the task of choosing the appropriate performance measure is up to the finance practitioner, not the statistician. Our aim is to provide a reliable inference method in case the comparison of two Sharpe ratios is deemed of interest.

**2. The problem**

We use the same notation as [Jobson and Korkie \(1981\)](#) and [Mommel \(2003\)](#). There are two investment strategies  $i$  and  $n$  whose excess returns over a given benchmark at time  $t$  are  $r_{ti}$  and  $r_{tn}$ , respectively. Typically, the benchmark is the risk-free rate but it could also be something else, such as a stock index.<sup>4</sup> A total of  $T$  return pairs  $(r_{1i}, r_{1n})', \dots, (r_{Ti}, r_{Tn})'$  are observed. It is assumed that these observations constitute a strictly stationary time series so that, in particular, the bivariate return distribution does not change over time. This distribution has mean vector  $\mu$  and covariance matrix  $\Sigma$  given by

$$\mu = \begin{pmatrix} \mu_i \\ \mu_n \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \sigma_i^2 & \sigma_{in} \\ \sigma_{in} & \sigma_n^2 \end{pmatrix}.$$

The usual sample means and sample variances of the observed returns are denoted by  $\hat{\mu}_i, \hat{\mu}_n$  and  $\hat{\sigma}_i^2, \hat{\sigma}_n^2$ , respectively. The difference between the two Sharpe ratios is given by

$$\Delta = \text{Sh}_i - \text{Sh}_n = \frac{\mu_i}{\sigma_i} - \frac{\mu_n}{\sigma_n}$$

and its estimator is

$$\hat{\Delta} = \widehat{\text{Sh}}_i - \widehat{\text{Sh}}_n = \frac{\hat{\mu}_i}{\hat{\sigma}_i} - \frac{\hat{\mu}_n}{\sigma_n}.$$

Furthermore, let  $u = (\mu_i, \mu_n, \sigma_i^2, \sigma_n^2)'$  and  $\hat{u} = (\hat{\mu}_i, \hat{\mu}_n, \hat{\sigma}_i^2, \hat{\sigma}_n^2)'$ . [Mommel \(2003\)](#) computes a standard error for  $\hat{\Delta}$  based on the relation

$$\sqrt{T}(\hat{u} - u) \xrightarrow{d} N(0; \Omega),$$

where  $\xrightarrow{d}$  denotes convergence in distribution, and an application of the delta method. However, just like [Jobson and Korkie \(1981\)](#), he uses a formula for  $\Omega$  that crucially relies on i.i.d. return data from a bivariate normal distribution, namely he assumes

$$\Omega = \begin{pmatrix} \sigma_i^2 & \sigma_{in} & 0 & 0 \\ \sigma_{in} & \sigma_n^2 & 0 & 0 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_{in}^2 \\ 0 & 0 & 2\sigma_{in}^2 & 2\sigma_n^4 \end{pmatrix}.$$

This formula is no longer valid if the distribution is non-normal or if the observations are correlated over time. To give just two examples, consider data that are i.i.d. but not necessarily normal. First, the entry in the lower right corner of  $\Omega$  is given by  $E[(r_{1n} - \mu_n)^4] - \sigma_n^4$  instead of by  $2\sigma_n^4$ . Second, the asymptotic covariance between  $\mu_n$  and  $\sigma_n^2$ , say, is in general not equal to zero.<sup>5</sup> To give another example, consider data from a stationary time series. Then the entry in the upper left corner is given by  $\sigma_i^2 + 2 \sum_{t=1}^{\infty} \text{Cov}(r_{1i}, r_{1+t|i})$  instead of by simply  $\sigma_i^2$ .

**3. Solutions**

We find it somewhat more convenient to work with the uncentered second moments. So let  $\gamma_i = E(r_{1i}^2)$  and  $\gamma_n = E(r_{1n}^2)$ . Their sample counterparts are denoted by  $\hat{\gamma}_i$  and  $\hat{\gamma}_n$ , respectively. Furthermore, let  $v = (\mu_i, \mu_n, \gamma_i, \gamma_n)'$  and  $\hat{v} = (\hat{\mu}_i, \hat{\mu}_n, \hat{\gamma}_i, \hat{\gamma}_n)'$ . This allows us to write

$$\Delta = f(v) \quad \text{and} \quad \hat{\Delta} = f(\hat{v}) \tag{1}$$

with

$$f(a, b, c, d) = \frac{a}{\sqrt{c - a^2}} - \frac{b}{\sqrt{d - b^2}}. \tag{2}$$

We assume that

$$\sqrt{T}(\hat{v} - v) \xrightarrow{d} N(0; \Psi), \tag{3}$$

<sup>4</sup> Strictly speaking, when the benchmark is a stock index, say, rather than the risk-free rate, one should speak of the Information ratio rather than the Sharpe ratio.

<sup>5</sup> For example, consider data from a Poisson distribution, in which case  $\hat{\mu}$  and  $\hat{\sigma}^2$  estimate the same parameter.

where  $\Psi$  is an unknown symmetric positive semi-definite matrix. This relation holds under mild regularity conditions. For example, when the data are assumed i.i.d., it is sufficient to have both  $E(r_{1t}^4)$  and  $E(r_{2t}^4)$  finite. In the time series case it is sufficient to have finite  $4+\delta$  moments, where  $\delta$  is some small positive constant, together with an appropriate mixing condition; for example, see Andrews (1991). The delta method then implies

$$\sqrt{T}(\hat{\Delta}-\Delta) \xrightarrow{d} N(0; \nabla' f(v)\Psi \nabla f(v)) \tag{4}$$

with

$$\nabla' f(a, b, c, d) = \left( \frac{c}{(c-a^2)^{1.5}}, -\frac{d}{(d-b^2)^{1.5}}, -\frac{1}{2} \frac{a}{(c-a^2)^{1.5}}, \frac{1}{2} \frac{b}{(d-b^2)^{1.5}} \right).$$

Now, if a consistent estimator  $\hat{\Psi}$ ; of  $\Psi$  is available, then a standard error for  $\hat{\Delta}$  is given by

$$s(\hat{\Delta}) = \sqrt{\frac{\nabla' f(\hat{v}) \hat{\Psi} \nabla f(\hat{v})}{T}}. \tag{5}$$

### 3.1. HAC inference

As is well-known, the limiting covariance matrix in Eq. (3) is given by

$$\Psi = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{s=1}^T \sum_{t=1}^T E[y_s y'_t], \quad \text{with } y'_t = (r_{1t} - \mu_1, r_{2t} - \mu_2, r_{1t}^2 - \gamma_1, r_{2t}^2 - \gamma_2).$$

By change of variables, the limit can be alternatively expressed as

$$\Psi = \lim_{T \rightarrow \infty} \Psi_T, \quad \text{with } \Psi_T = \sum_{j=-T+1}^{T-1} \Gamma_T(j), \quad \text{where}$$

$$\Gamma_T(j) = \begin{cases} \frac{1}{T} \sum_{t=j+1}^T E[y_t y'_{t-j}] & \text{for } j \geq 0 \\ \frac{1}{T} \sum_{t=-j+1}^T E[y_{t+j} y'_t] & \text{for } j < 0 \end{cases}.$$

The standard method to come up with a consistent estimator  $\hat{\Psi} = \hat{\Psi}_T$  is to use heteroskedasticity and autocorrelation robust (HAC) kernel estimation; for example, see Andrews (1991) and Andrews and Monahan (1992). In practice this involves choosing a real-valued kernel function  $k(\cdot)$  and a bandwidth  $S_T$ . The kernel  $k(\cdot)$  typically satisfies the three conditions  $k(0)=1$ ,  $k(\cdot)$  is continuous at 0, and  $\lim_{x \rightarrow \pm\infty} k(x)=0$ . The kernel estimate for  $\Psi$  is then given by

$$\hat{\Psi} = \hat{\Psi}_T = \frac{T}{T-4} \sum_{j=-T+1}^{T-1} k\left(\frac{j}{S_T}\right) \hat{\Gamma}_T(j), \quad \text{where}$$

$$\hat{\Gamma}_T(j) = \begin{cases} \frac{1}{T} \sum_{t=j+1}^T \hat{y}_t \hat{y}'_{t-j} & \text{for } j \geq 0 \\ \frac{1}{T} \sum_{t=-j+1}^T \hat{y}_{t+j} \hat{y}'_t & \text{for } j < 0 \end{cases}, \quad \text{with } \hat{y}'_t = (r_{1t} - \hat{\mu}_1, r_{2t} - \hat{\mu}_2, r_{1t}^2 - \hat{\gamma}_1, r_{2t}^2 - \hat{\gamma}_2).$$

The factor  $T/(T-4)$  is a small-sample degrees of freedom adjustment that is introduced to offset the effect of the estimation of the 4-vector  $v$  in the computation of the  $\hat{\Gamma}_T(j)$ , that is, the use of the  $\hat{y}_t$  rather than the  $y_t$ .

An important feature of a kernel  $k(\cdot)$  is its characteristic exponent  $1 \leq q \leq \infty$ , determined by the smoothness of the kernel at the origin. Note that the bigger  $q$ , the smaller is the asymptotic bias of a kernel variance estimator; on the other hand, only kernels with  $q \leq 2$  yield estimates that are guaranteed to be positive semi-definite in finite samples. Most of the commonly used kernels have  $q=2$ , such as the Parzen, Tukey–Hanning, and Quadratic-Spectral (QS) kernels, but exceptions do exist. For example, the Bartlett kernel has  $q=1$  and the Truncated kernel has  $q=\infty$ . For a broader discussion on this issue, see Andrews (1991) again.

Once a particular kernel  $k(\cdot)$  has been chosen for application, one must pick the bandwidth  $S_T$ . Several automatic methods, based on various asymptotic optimality criteria, are available to this end; for example, see Andrews (1991) and Newey and West (1994).

Finally, given the kernel estimator  $\hat{\Psi}$ , the standard error  $s(\hat{\Delta})$  is obtained as in Eq. (5) and then combined with the asymptotic normality of Eq. (4) to make HAC inference as follows.

A two-sided  $p$ -value for the null hypothesis  $H_0: \Delta=0$  is given by

$$\hat{p} = 2\Phi\left(-\frac{|\hat{\Delta}|}{s(\hat{\Delta})}\right)$$

where  $\Phi(\cdot)$  denotes the c.d.f. of the standard normal distribution. Alternatively, a  $1-\alpha$  confidence interval for  $\Delta$  is given by

$$\hat{\Delta} \pm z_{1-\alpha/2} s(\hat{\Delta})$$

where  $z_\lambda$  denotes the  $\lambda$  quantile of the standard normal distribution, that is,  $\Phi(z_\lambda)=\lambda$ .

It is, however, well-known that such HAC inference is often liberal when sample sizes are small to moderate. This means hypothesis tests tend to reject a true null hypothesis too often compared to the nominal significance level and confidence intervals tend to undercover; for example, see Andrews (1991), Andrews and Monahan (1992), and Romano and Wolf (2006).

**Remark 3.1.** Lo (2002) discusses inference for a single Sharpe ratio. The method he presents in the section titled “IID Returns” corresponds to Jobson and Korkie (1981) and Memmel (2003), since it actually assumes i.i.d. normal data. The method he presents in the section titled “Non-IID Returns” corresponds to the HAC inference of this subsection.

Opdyke (2007) discusses both inferences for a single Sharpe ratio and for the difference of two Sharpe ratios. He first considers the case of general i.i.d. data (i.e., not necessarily normal) and corrects the formulae for the limiting variances of Lo (2002, Section “IID Returns”) and of Memmel (2003), respectively, which assume normality. He also considers the case of general stationary data (i.e., time series). However, his formulae for the time series case are incorrect, since they are equivalent to the formulae for the i.i.d. case. For example, the limiting variance in Eq. (6), for the case of general stationary data, is equivalent to the limiting variance in Eq. (3), for general i.i.d. data. The problem is that in the case of general stationary data, simple and easily-implemented formulae do not exist.

### 3.2. Bootstrap inference

There is an extensive literature demonstrating the improved inference accuracy of the studentized bootstrap over ‘standard’ inference based on asymptotic normality; see Hall (1992) for i.i.d. data and Lahiri (2003) for time series data. Very general results are available for parameters of interests which are smooth functions of means. Our parameter of interest,  $\Delta$ , fits this bill; see Eqs. (1) and (2).

Arguably, the regularity conditions used by Lahiri (2003, Section 6.5) in the time series case are rather strong (and too strong for most financial applications); for example, they assume  $35+\delta$  finite moments (where  $\delta$  is some small number) and certain restrictions on the dependence structure.<sup>6</sup> However, it should be pointed out that these conditions are *sufficient* only to prove the very complex underlying mathematics and not necessary. Even when these conditions do not hold, the studentized bootstrap typically continues to outperform ‘standard’ inference; e.g., see Section 4. To avoid any confusion, it should also be pointed that these strong regularity conditions are only needed to prove the superiority of the studentized bootstrap. In terms of first-order consistency it does not really need stronger sufficient conditions than ‘standard’ inference.

We propose to test  $H_0: \Delta=0$  by inverting a bootstrap confidence interval. That is, one constructs a two-sided bootstrap confidence interval with nominal level  $1-\alpha$  for  $\Delta$ . If this interval does not contain zero, then  $H_0$  is rejected at nominal level  $\alpha$ . The advantage of this ‘indirect’ approach is that one can simply resample from the observed data. If one wanted to carry out a ‘direct’ bootstrap test, one would have to resample from a probability distribution that satisfied the constraint of the null hypothesis, that is, from some modified data where the two empirical Sharpe ratios were exactly equal; e.g., see Politis et al. (1999, Section 1.8).

In particular, we propose to construct a symmetric studentized bootstrap confidence interval. To this end the two-sided distribution function of the studentized statistic is approximated via the bootstrap as follows:

$$\mathcal{L}\left(\frac{|\hat{\Delta}-\Delta|}{s(\hat{\Delta})}\right) \approx \mathcal{L}\left(\frac{|\hat{\Delta}^*-\hat{\Delta}|}{s(\hat{\Delta}^*)}\right). \tag{6}$$

In this notation,  $\Delta$  is true difference between the Sharpe ratios,  $\hat{\Delta}$  is the estimated difference computed from the original data,  $s(\hat{\Delta})$  is a standard error for  $\hat{\Delta}$  (also computed from the original data),  $\hat{\Delta}^*$  is the estimated difference computed from bootstrap data, and  $s(\hat{\Delta}^*)$  is a standard error for  $\hat{\Delta}^*$  (also computed from bootstrap data). Finally,  $\mathcal{L}(X)$  denotes the distribution of the random variable  $X$ .

Letting  $z_{|\cdot|,\lambda}^*$  be a  $\lambda$  quantile of  $\mathcal{L}(|\hat{\Delta}^*-\hat{\Delta}|/s(\hat{\Delta}^*))$ , a bootstrap  $1-\alpha$  confidence interval for  $\Delta$  is then given by

$$\hat{\Delta} \pm z_{|\cdot|,1-\alpha}^* s(\hat{\Delta}). \tag{7}$$

The point is that when data are heavy-tailed or of time series nature, then  $z_{|\cdot|,1-\alpha}^*$  will typically be somewhat larger than  $z_{1-\alpha/2}$  for small to moderate samples, resulting in more conservative inference compared to the HAC methods of the previous subsection.

We are left to specify (i) how the bootstrap data are to be generated and (ii) how the standard errors  $s(\hat{\Delta})$  and  $s(\hat{\Delta}^*)$  are to be computed. For this, it is useful to distinguish between i.i.d. data and time series data. The first case, i.i.d. data, is included mainly for

<sup>6</sup> The conditions are too lengthy to be reproduced here.

completeness of the exposition. It is well-known that financial returns are generally not i.i.d. Even when the autocorrelation of the returns is negligible (which often happens with the stock and mutual fund returns), there usually exists autocorrelation of the squared returns, that is, volatility clustering. We therefore recommend to always use the bootstrap method for time series data in practice.

3.2.1. I.I.D. data

To generate bootstrap data, one simply uses Efron's (1979) bootstrap, resampling individual pairs from the observed pairs  $(r_{ti}, r_{tm})'$ ,  $t = 1, \dots, T$ , with replacement. The standard error  $s(\hat{\Delta})$  is computed as in Eq. (5). Since the data are i.i.d., one takes for  $\hat{\Psi}$  here simply the sample covariance matrix of the vectors  $(r_{ti}, r_{tm}, r_{ti}^2, r_{tm}^2)'$ ,  $t = 1, \dots, T$ . The standard error  $s(\hat{\Delta}^*)$  is computed in exactly the same fashion but from the bootstrap data instead of the original data. To be more specific, denote the  $t$ th return pair of the bootstrap sample by  $(r_{ti}^*, r_{tm}^*)'$ . Then one takes for  $\hat{\Psi}^*$  the sample covariance matrix of the vectors  $(r_{ti}^*, r_{tm}^*, r_{ti}^{*2}, r_{tm}^{*2})'$ ,  $t = 1, \dots, T$ . Furthermore, the estimator of  $v = (\mu_i, \mu_n, \gamma_i, \gamma_n)'$  based on the bootstrap data is denoted by  $\hat{v}^* = (\hat{\mu}_i^*, \hat{\mu}_n^*, \hat{\gamma}_i^*, \hat{\gamma}_n^*)'$ . Finally, the bootstrap standard error for  $\hat{\Delta}^*$  is given by

$$s(\hat{\Delta}^*) = \sqrt{\frac{\nabla' f(\hat{v}^*) \hat{\Psi}^* \nabla f(\hat{v}^*)}{T}} \tag{8}$$

3.2.2. Time series data

The application of the studentized bootstrap is somewhat more involved when the data are of time series nature. To generate bootstrap data, we use the circular block bootstrap of Politis and Romano (1992), resampling now blocks of pairs from the observed pairs  $(r_{ti}, r_{tm})'$ ,  $t = 1, \dots, T$ , with replacement.<sup>7</sup> These blocks have a fixed size  $b \geq 1$ . The standard error  $s(\hat{\Delta})$  is computed as in Eq. (5). The estimator  $\hat{\Psi}$  is obtained via kernel estimation; in particular we propose the prewhitened QS kernel of Andrews and Monahan (1992).<sup>8</sup> The standard error  $s(\hat{\Delta}^*)$  is the 'natural' standard error computed from the bootstrap data, making use of the special block dependence structure; see Götze and Künsch (1996) for details. To be more specific, let  $l = \lfloor n/b \rfloor$ , where  $\lfloor \cdot \rfloor$  denotes the integer part. Again, the estimator of  $v = (\mu_i, \mu_n, \gamma_i, \gamma_n)'$  based on the bootstrap data is denoted by  $\hat{v}^* = (\hat{\mu}_i^*, \hat{\mu}_n^*, \hat{\gamma}_i^*, \hat{\gamma}_n^*)'$ . Then define

$$y_t^* = \left( r_{ti}^* - \hat{\mu}_i^*, r_{tm}^* - \hat{\mu}_n^*, r_{ti}^{*2} - \hat{\gamma}_i^*, r_{tm}^{*2} - \hat{\gamma}_n^* \right) \quad t = 1, \dots, T \quad \zeta_j = \frac{1}{\sqrt{b}} \sum_{t=1}^b y_{(j-1)b+t}^* \quad t = 1, \dots, l$$

and

$$\hat{\Psi}^* = \frac{1}{l} \sum_{j=1}^l \zeta_j \zeta_j'$$

With this more general definition<sup>9</sup> of  $\hat{\Psi}^*$ , the bootstrap standard error for  $\hat{\Delta}^*$  is again given by formula (8).

An application of the studentized circular block bootstrap requires a choice of the block size  $b$ . To this end, we suggest to use a calibration method, a concept dating back to Loh (1987). One can think of the actual coverage level  $1 - \lambda$  of a block bootstrap confidence interval as a function of the block size  $b$ , conditional on the underlying probability mechanism  $P$  which generated the bivariate time series of returns, the nominal confidence level  $1 - \alpha$ , and the sample size  $T$ . The idea is now to adjust the 'input'  $b$  in order to obtain the actual coverage level close to the desired one. Hence, one can consider the block size calibration function  $g: b \rightarrow 1 - \lambda$ . If  $g(\cdot)$  were known, one could construct an 'optimal' confidence interval by finding  $b$  that minimizes  $|g(b) - (1 - \alpha)|$  and then use  $\hat{b}$  as the block size of the time series bootstrap; note that  $|g(b) - (1 - \alpha)| = 0$  may not always have a solution.

Of course, the function  $g(\cdot)$  depends on the underlying probability mechanism  $P$  and is therefore unknown. We now propose a bootstrap method to estimate it. The idea is that in principle we could simulate  $g(\cdot)$  if  $P$  were known by generating data of size  $T$  according to  $P$  and by computing confidence intervals for  $\Delta$  for a number of different block sizes  $b$ . This process is then repeated many times and for a given  $b$ , one estimates  $g(b)$  as the fraction of the corresponding intervals that contain the true parameter. The method we propose is identical except that  $P$  is replaced by an estimate  $\hat{P}$  and that the true parameter  $\Delta$  is replaced by the 'pseudo' parameter  $\hat{\Delta}$ .

**Algorithm 3.1 (Choice of the block size).**

1. Fit a semi-parametric model  $\hat{P}$  to the observed data  $(r_{1i}, r_{1n})', \dots, (r_{Ti}, r_{Tn})'$ .
2. Fix a selection of reasonable block sizes  $b$ .
3. Generate  $K$  pseudo sequences  $(r_{1i}^*, r_{1n}^*)', \dots, (r_{Ti}^*, r_{Tn}^*)'$ ,  $k = 1, \dots, K$ , according to  $\hat{P}$ . For each sequence,  $k = 1, \dots, K$ , and for each  $b$ , compute a confidence interval  $CI_{k,b}$  with nominal level  $1 - \alpha$  for  $\hat{\Delta}$ .
4. Compute  $g(b) = \#\{\hat{\Delta} \in CI_{k,b}\} / K$ .
5. Find the value  $b$  that minimizes  $|\hat{g}(b) - (1 - \alpha)|$ .

<sup>7</sup> The motivation for using the circular block bootstrap instead of the moving blocks bootstrap of Künsch (1989) is to avoid the 'edge effects' of the latter; see Romano and Wolf (2006, Section 4).

<sup>8</sup> We have found that the prewhitened Parzen kernel, which is defined analogously, yields very similar performance.

<sup>9</sup> Note that for the special case  $b=1$ , this definition results in the sample covariance matrix of the bootstrap data  $(r_{ti}^*, r_{tm}^*, r_{ti}^{*2}, r_{tm}^{*2})'$ ,  $t = 1, \dots, T$ , used for i.i.d. data.

Of course, the question remains which semi-parametric model to fit to the observed return data. When using monthly data, we recommend to simply use a VAR model in conjunction with time series bootstrapping the residuals.<sup>10</sup> If the data are sampled at finer intervals, such as daily data, one might want to use a bivariate GARCH model instead.

Next, one might ask what is a selection of reasonable block sizes? The answer is any selection that contains a  $b$  with  $\hat{g}(b)$  very close to  $1 - \alpha$ . If nothing else, this can always be determined by trial and error. In our experience,  $\hat{g}(\cdot)$  is typically monotonically decreasing in  $b$ .

So if one starts with  $b_{\text{low}}=1$  and a  $b_{\text{up}}$  'sufficiently' large, one is left to specify some suitable grid between those two values. In our experience, again, for a sample size of  $T=120$ , the choices  $b_{\text{low}}=1$  and  $b_{\text{up}}=10$  usually suffice. In that case the final selection  $\{1, 2, 4, 6, 8, 10\}$  should be fine, as  $\hat{g}(\cdot)$  does not tend to decrease very fast in  $b$ .

Finally, how large should  $K$  be chosen in application to real data? The answer is as large as possible, given the computational resources.  $K=5000$  will certainly suffice for all practical purposes, while  $K=1000$  should be the lower limit.

**Remark 3.2.** As outlined above, a two-sided test for  $H_0: \Delta=0$  at significance level  $\alpha$  can be carried out by constructing a bootstrap confidence interval with confidence level  $1 - \alpha$ . The test rejects if zero is not contained in the interval. At times, it might be more desirable to obtain a  $p$ -value. In principle, such a  $p$ -value could be computed by 'trial and error' as the smallest  $\alpha$  for which the corresponding  $1 - \alpha$  confidence interval does not contain zero. However, such a procedure is rather cumbersome. Fortunately, there exists a shortcut that allows for an equivalent 'direct' computation of such a  $p$ -value. Denote the original studentized test statistic by  $d$ , that is,

$$d = \frac{|\hat{\Delta}|}{s(\hat{\Delta})}$$

and denote the centered studentized statistic computed from the  $m$ th bootstrap sample by  $d^{*,m}$ ,  $m=1, \dots, M$ , that is,

$$\tilde{d}^{*,m} = \frac{|\hat{\Delta}^{*,m} - \hat{\Delta}|}{s(\hat{\Delta}^{*,m})},$$

where  $M$  is the number of bootstrap resamples. Then the  $p$ -value is computed as

$$\text{PV} = \frac{\{ \tilde{d}^{*,m} \geq d \} + 1}{M + 1}. \quad (9)$$

**Remark 3.3.** As far as we know, there have been two previous proposals to use bootstrap inference for Sharpe ratios, but both are somewhat limited.

Vinod and Morey (1999) discuss inference for the difference of two Sharpe ratios. However, they only employ Efron's (1979) i.i.d. bootstrap, so their approach would not work for time series data. Moreover, the way they studentize in the bootstrap world is incorrect, as they use a common standard error for all bootstrap statistics (instead, one must compute an individual standard error from each bootstrap data set, as described before).

Scherer (2004) discusses inference for a single Sharpe ratio, but his approach could be easily extended to inference for a difference of two Sharpe ratios. Unlike us, he employs a non-studentized bootstrap for both i.i.d. and time series data. The problem is that a non-studentized bootstrap does not provide improved inference accuracy compared to 'standard' inference based on asymptotic normality; again see Hall (1992) and Lahiri (2003). Scherer (2004) addresses this problem for i.i.d. data by employing a double bootstrap (which also provides improved inference accuracy; Hall, 1992), but he does not address it for time series data. Moreover, his time series bootstrap is of parametric nature, based on an AR(1) model with i.i.d. errors, and would therefore not be valid in general.

Incidentally, the asymptotically valid approaches detailed in this paper, HAC inference and the studentized bootstrap, can be easily modified to make inference for a single Sharpe ratio. The details are straightforward and left to the reader.

#### 4. Simulation study

The purpose of this section is to shed some light on the finite sample performance of the various methods via some (necessarily limited) simulations. We compute empirical rejection probabilities under the null, based on 5000 simulations per scenario. The nominal levels considered are  $\alpha=0.01, 0.5, 0.1$ . All bootstrap  $p$ -values are computed as in Eq. (9), employing  $M=499$ . The sample size is  $T=120$  always.<sup>11</sup>

<sup>10</sup> At this point we opt for the stationary bootstrap of Politis and Romano (1994), since it is quite insensitive to the choice of the average block size. The motivation for time series bootstrapping the residuals is to account for some possible 'left over' non-linear dependence not captured by the linear VAR model.

<sup>11</sup> Many empirical applications use ten years of monthly data.



#### 4.1. Competing methods

The following methods are included in the study:

- (JKM) The test of [Jobson and Korkie \(1981\)](#), using the corrected version of [Mommel \(2003\)](#).
- (HAC) The HAC test of Subsection 3.1 based on the QS kernel with automatic bandwidth selection of [Andrews \(1991\)](#).
- (HAC<sub>PW</sub>) The HAC test of Subsection 3.1 based on the prewhitened QS kernel with automatic bandwidth selection of [Andrews and Monahan \(1992\)](#).
- (Boot-IID) The bootstrap method of Subsubsection 3.2.1.
- (Boot-TS) The bootstrap method of Subsubsection 3.2.2. We use Algorithm 3.1 to pick a data-dependent block size from the input block sizes  $b \in \{1, 2, 4, 6, 8, 10\}$ . The semi-parametric model used is a VAR(1) model in conjunction with bootstrapping the residuals. For the latter we employ the stationary bootstrap of [Politis and Romano \(1994\)](#) with an average block size of 5.

#### 4.2. Data generating processes

In all scenarios, we want the null hypothesis of equal Sharpe ratios to be true. This is easiest achieved if the two marginal return processes are identical.

It is natural to start with i.i.d. bivariate normal data with equal mean 1 and equal variance 1. The within-pair correlation is chosen as  $\rho=0.5$ , which seems a reasonable number for many applications. This DGP is denoted by Normal-IID.

We then relax the strict i.i.d. normal assumption in various dimensions.

First, we keep the i.i.d. assumption but allow for heavy tails. To this end, we use bivariate  $t_6$  data, shifted to have equal mean 1 and standardized to have common variance 1. The within-pair correlation is  $\rho=0.5$  again. This DGP is denoted by  $t_6$ -IID.

Next, we consider an uncorrelated process but with correlations in the squared returns, as is typical for stock returns. The standard way to model this is via a bivariate GARCH(1,1) model. In particular, we use the bivariate *diagonal-vech* model dating back to [Bollerslev et al. \(1988\)](#). Let  $\tilde{r}_{it} = r_{it} - \mu_i$ ,  $\tilde{r}_{mt} = r_{mt} - \mu_m$ , and denote by  $\Omega_{t-1}$  the conditioning information available at time  $t-1$ . Then the diagonal-vech model is defined by

$$\begin{aligned} E(\tilde{r}_{it} | \Omega_{t-1}) &= 0 & E(\tilde{r}_{mt} | \Omega_{t-1}) &= 0 & \text{Cov}(\tilde{r}_{it}, \tilde{r}_{mt} | \Omega_{t-1}) &= h_{tin} \\ & & & & &= c_{in} + a_{in} \tilde{r}_{(t-1)i} \tilde{r}_{(t-1)m} + b_{in} h_{(t-1)in}. \end{aligned}$$

In other words, the conditional (co)variances depend only on their own lags and the lags of the corresponding (cross) products. We use the following coefficient matrices:

$$C = \begin{pmatrix} 0.15 & 0.13 \\ 0.13 & 0.15 \end{pmatrix} \quad A = \begin{pmatrix} 0.075 & 0.050 \\ 0.050 & 0.075 \end{pmatrix} \quad B = \begin{pmatrix} 0.90 & 0.89 \\ 0.90 & 0.89 \end{pmatrix}.$$

These matrices are inspired by the bivariate estimation results based on weekly returns on a broad U.S. market index and a broad U.K. market index.<sup>12</sup> However, all three diagonals are forced to be equal to get identical individual return processes; see [Ledoit et al. \(2003, Table 2\)](#).

The first variant of the GARCH model uses i.i.d. bivariate standard normal innovations to recursively generate the series  $\tilde{r}_t = (\tilde{r}_{it}, \tilde{r}_{mt})'$ . At the end, we add a global mean, that is,  $r_t = \tilde{r}_t + \mu$ , where  $\mu$  is chosen as  $\mu = (16.5/52, 16.5/52)'$ . Again this choice is inspired by the previously mentioned estimation results, forcing  $\mu_i = \mu_m$  to get identical individual return processes; see [Ledoit et al. \(2003, Table 1\)](#). This DGP is denoted by Normal-GARCH.

The second variant of the GARCH model uses i.i.d. bivariate  $t_6$  innovations instead (standardized to have common variance equal to 1, and covariance equal to 0).<sup>13</sup> Everything else is equal. This DGP is denoted by  $t_6$ -GARCH.

Finally, we also consider correlated processes. To this end, we return to the two i.i.d. DGPs Normal-IID and  $t_6$ -IID, respectively, but add some mild autocorrelation to the individual return series via an AR(1) structure with AR coefficient  $\phi=0.2$ .<sup>14</sup> This then corresponds to a VAR(1) model with bivariate normal or (standardized)  $t_6$  innovations. The resulting two DGPs are denoted by Normal-VAR and  $t_6$ -VAR, respectively.

## 5. Results

The results are presented in [Table 1](#) and summarized as follows:

- JKM works well for i.i.d. bivariate normal data but is not robust against fat tails or time series effects, where it becomes liberal.
- HAC inference, while asymptotically consistent, is often liberal in finite samples. This finding is consistent with many previous studies; e.g., see [Romano and Wolf \(2006\)](#) and the references therein.

<sup>12</sup> We use estimation results based on weekly returns, since generally there are very few GARCH effects at monthly or longer return horizons. With weekly data,  $T=120$  corresponds to a data window of slightly over two years.

<sup>13</sup> There is ample evidence that the innovations of GARCH processes tend to have tails heavier than the normal distribution; e.g., see [Kuester et al. \(2006\)](#) and the references therein.

<sup>14</sup> For example, a first-order autocorrelation around 0.2 is quite typical for monthly hedge fund returns.

**Table 1**

Empirical rejection probabilities (in percent) for various data generating processes (DGPs) and inference methods; see Section 4 for a description

DGP	JKM	HAC	HAC <sub>PW</sub>	Boot-IID	Boot-TS
<i>Nominal level <math>\alpha=1\%</math></i>					
Normal-IID	1.2	1.2	1.2	1.1	1.0
$t_6$ -IID	3.5	1.9	2.1	1.4	1.3
Normal-GARCH	1.7	1.8	1.8	1.5	1.1
$t_6$ -GARCH	1.8	2.0	2.0	1.6	1.2
Normal-VAR	2.5	2.2	1.8	2.7	1.2
$t_6$ -VAR	6.4	2.6	2.2	1.8	1.1
<i>Nominal level <math>\alpha=5\%</math></i>					
Normal-IID	5.0	5.3	5.4	4.9	4.8
$t_6$ -IID	10.7	6.7	6.9	5.2	5.0
Normal-GARCH	7.2	7.1	7.2	6.0	5.5
$t_6$ -GARCH	7.4	7.7	7.5	6.9	5.7
Normal-VAR	9.5	6.9	6.1	8.5	5.0
$t_6$ -VAR	14.5	7.9	7.3	7.3	5.1
<i>Nominal level <math>\alpha=10\%</math></i>					
Normal-IID	10.3	10.3	10.7	10.1	9.6
$t_6$ -IID	17.9	12.4	12.5	10.3	9.9
Normal-GARCH	12.8	12.5	12.3	12.4	10.5
$t_6$ -GARCH	13.7	13.3	13.1	13.1	11.1
Normal-VAR	15.6	12.4	10.8	15.6	9.7
$t_6$ -VAR	22.5	13.3	12.0	13.3	9.8

For each DGP, the null hypothesis of equal Sharpe ratios is true and so the empirical rejection probabilities should be compared to the nominal level of the test, given by  $\alpha$ . We consider three values of  $\alpha$ , namely  $\alpha=1\%$ ,  $5\%$  and  $10\%$ . All empirical rejection probabilities are computed from 5000 repetitions of the underlying DGP, and the same set of repetitions is shared by all inference methods.

**Table 2**

Summary sample statistics for monthly log returns in excess of the risk-free rate: mean, standard deviation, Sharpe ratio, and first-order autocorrelation

Fund	$\bar{r}$	$s$	$\hat{S}_h$	$\hat{\phi}$
Fidelity	0.511	4.760	0.108	-0.010
Fidelity Aggressive Growth	0.098	9.161	0.011	0.090
Coast Enhanced Income	0.245	0.168	1.461	0.152
JMG Capital Partners	1.228	1.211	1.014	0.435

- Boot-IID works well for i.i.d. data but is liberal for time series data.
- Boot-TS works well both for i.i.d. and time series data.

**Remark 4.1.** We also included HAC and HAC<sub>PW</sub> based on the (prewhitened) Parzen kernel instead of the (prewhitened) QS kernel. The results were virtually identical and are therefore not reported. Since the Parzen kernel has a finite support while the QS kernel does not, it is somewhat more convenient to implement; e.g., see [Andrews \(1991\)](#).

## 6. Empirical applications

As a brief illustration, we consider two applications to investment funds. In each case, we want to test the null hypothesis of equality of the Sharpe ratios of the two funds being compared.

The first application deals with mutual funds. The selected funds are Fidelity (FFIDX), a ‘large blend’ fund, and Fidelity Aggressive Growth (FDEGX), a ‘mid-cap growth’ fund. The data were obtained from Yahoo! Finance.<sup>15</sup>

The second application deals with hedge funds. The selected funds are Coast Enhanced Income and JMG Capital Partners. The data were obtained from the CISDM database; see [Romano et al. \(2008, Section 9\)](#).

In both applications, we use monthly log returns in excess of the risk-free rate. The return period is 01/1994 until 12/2003, so  $T=120$ . [Table 2](#) provides some relevant summary statistics. Note that all returns are in percentages and that none of the statistics are annualized.

[Table 3](#) presents the corresponding  $p$ -values of the five methods previously considered in the simulation study. Boot-TS uses a data-dependent choice of block size based on Algorithm 3.1. The semi-parametric model is a VAR(1) model in conjunction with bootstrapping the residuals. For the latter we employ the stationary bootstrap of [Politis and Romano \(1994\)](#) with an average block

<sup>15</sup> We use close prices adjusted for dividends and stock splits.



**Table 3**

*p*-values (in percent) for various inference methods; see Section 4 for a description

Data	JKM	HAC	HAC <sub>PW</sub>	Boot-IID	Boot-TS
Mutual funds	3.9	6.3	6.7	4.4	9.2
Hedge funds	1.0	14.7	25.4	5.8	29.4

The data set 'mutual funds' corresponds to the top two funds of Table 2; the data set 'hedge funds' corresponds to the bottom two funds of Table 2. All *p*-values are for the two-sided test of equal Sharpe ratios.

size of 5. The nominal confidence level is  $1 - \alpha = 0.95$  and the set of input block sizes is  $\{1, 2, 4, 6, 8, 10\}$ . The two estimated calibration functions, based on  $K = 5000$  pseudo sequences, are displayed in Fig. 1. As a result, the estimated optimal block sizes are  $\hat{b} = 4$  for the mutual funds application and  $\hat{b} = 6$  for the hedge funds application.

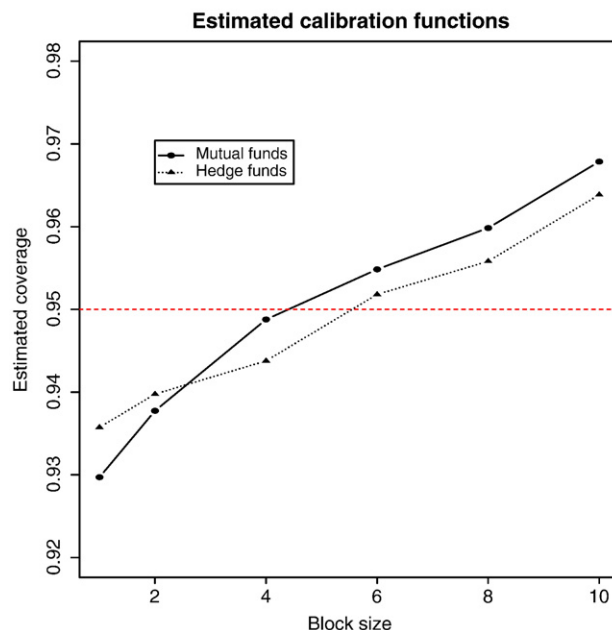
The bootstrap *p*-values are computed as in Eq. (9), employing  $M = 4999$ . In both applications, JKM results in a rejection of the null at significance level  $\alpha = 0.05$ , while HAC, HAC<sub>PW</sub>, and Boot-TS do not. Not surprisingly, given the noticeable autocorrelation of hedge fund returns, the differences are more pronounced for the second application. Boot-IID results in a rejection for the mutual funds data but not for the hedge fund data. However, as discussed previously, we recommend to always use Boot-TS with financial return data.

## 7. Conclusion

Testing for the equality of the Sharpe ratios of two investment strategies is an important tool for performance analysis. The current *status quo* method in the applied literature appears to be the test of Memmel (2003), which is a corrected version of the original proposal of Jobson and Korkie (1981). Unfortunately, this test is not robust against tails heavier than the normal distribution and time series characteristics. Since both effects are quite common with financial returns, the test should not be used.

We have discussed alternative inference methods which are robust. HAC inference uses kernel estimators to come up with consistent standard errors. The resulting inference works well with large samples but is often liberal for small to moderate sample sizes. In such applications, it is preferable to use a studentized time series bootstrap. Arguably, this procedure is quite complex to implement, but corresponding programming code is freely available at <http://www.iew.uzh.ch/chairs/wolf/team/wolf/publications/en.html>.

Finally, both the HAC inference and the studentized bootstrap detailed in this paper could be modified to make inference for (the difference of) various refinements to the Sharpe ratio recently proposed in the literature—e.g., see Ferruz and Vicente (2005) and Israelsen (2003, 2005)—as well as many other performance measures, such as the Information ratio, Jensen's alpha, or the Treynor ratio, to name just a few.



**Fig. 1.** Estimated calibration functions for the two empirical applications. The nominal level is  $1 - \alpha = 0.95$ . The resulting estimated optimal block sizes are  $b = 4$  for the mutual funds application and  $b = 6$  for the hedge funds application.

## Acknowledgment

We are grateful for two helpful reports from anonymous referees which have led to an improved presentation of the paper.

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